MODULATION EQUATIONS FOR A MIXTURE OF GAS BUBBLES IN AN INCOMPRESSIBLE LIQUID

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We consider the one-fluid flow of a monodispersed mixture of bubbles with an incompressible liquid. We derive a system of equations of motion which are intermediate between the known equilibrium and nonequilibrium models. The advantage of these equations is that they lead to separation between the average characteristics of the flow (the velocity, pressure, total energy of the mixture) and fast oscillations on the background of this average motion. In the absence of dissipation the equations for the average quantities reduce to the equations of motion of an inviscid, nonconducting gas.

1. Equations of Motion. The one-fluid motion of a monodispersed mixture of bubbles with an incompressible liquid is considered in the framework of the Iordanskii–Kogarko model [1-4]:

$$v_t - u_x = 0,\ u_t + p_x = 0; \tag{1.1}$$

$$RR_{tt} + 3R_t^2/2 = (p_2 - p - 2\sigma/R)/\rho_1 - 4\nu_1 R_t/R; \tag{1.2}$$

$$c_{2t} = 0,\ n_t = 0. \tag{1.3}$$

Here t is time; x, mass Lagrangian coordinate (see [5], for example); u, velocity of the mixture; $v = c_1/\rho_1 + c_2 v_2$, the specific volume of the mixture; c_i, the mass concentrations ($c_1 + c_2 = 1$, $c_2 = \alpha_2 v/v_2$); α_2 is the volumetric concentration of the gas phase; ρ_1, density of the liquid phase, which is assumed to be incompressible; $v_2 = n4\pi R^3/(3c_2)$, the specific volume of the gas phase; R, the radius of a bubble; σ, the surface tension; n, the number of bubbles per unit mass of the mixture; p, pressure of the mixture, which is assumed to be equal to the pressure of the liquid phase; $p_2 = p_2(v_2)$, the pressure of the gas phase; ν_1, the kinematic viscosity of the liquid phase. The mass Lagrangian coordinate x is introduced to shorten the equations. We note that the Rayleigh–Lamb equation (1.2), with (1.1) and (1.3) taken into account, can be written in the form

$$(\varepsilon + u^2/2 + n(E_r + E_\sigma))_t + (pu)_x = -8\nu_1 n E_r/R^2, \tag{1.2a}$$

where

$$\varepsilon = c_2\varepsilon_2(v_2),\ d\varepsilon_2/dv_2 = -p_2,\ E_r = 2\pi R^3\rho_1 R_t^2,\ E_\sigma = 4\pi R^2\sigma. \tag{1.4}$$

Equation (1.2) follows directly from (1.2a) after differentiation and use of (1.4). For a compressible liquid the Rayleigh–Lamb equation can be written in the form of a conservation law; see [6], for example.

We consider rapidly oscillating solutions of the system (1.1)-(1.3). This means that there are two characteristic scales of length in the flow: λ and L, the wavelengths of short and modulated waves, such that the parameter $\delta = \lambda/L$ is assumed to be small. A general method of studying solutions of this type was worked out for conservative systems by Whitham and associates [7]. In essence, this method is a special case of the averaging method of Krylov–Bogolyubov–Mitropol'skii. A discussion of asymptotic methods of constructing such singular solutions and their applications to hydrodynamics, nonlinear optics, etc., was given in [8].

2. Derivation of the Modulation Equations. We use (1.1)-(1.3) as the starting point. Let δ be a small parameter. Following [7], we introduce the slow variables $T = \delta t$, $X = \delta x$, and the fast variable (phase) $\theta = \Theta(T, X)/\delta$; we call $k = \theta_x = \Theta_X$ the local wave number and $\omega = -\theta_t = -\Theta_t$ the local frequency. We let $w = (R, p, u, c_2, n)$, and assume that $w = w(\theta, T,$

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X), where the dependence on the variable θ is assumed to be periodic. The period can be taken as 2π with no loss of generality, since if the period is τ_0 we can make the change of variable $\theta' = 2\pi\theta/\tau_0$. Using the derivative formulas

$$\frac{\partial}{\partial t} \to -\omega\frac{\partial}{\partial\theta} + \delta\frac{\partial}{\partial T},\ \frac{\partial}{\partial x} \to k\frac{\partial}{\partial\theta} + \delta\frac{\partial}{\partial X},$$

we obtain the following expanded set of equations from (1.1)-(1.3):

$$\delta(v_T - u_X) + (-v\omega - uk)_\theta = 0,\ \delta(u_T + p_X) + (-u\omega + pk)_\theta = 0,$$
$$\delta\{(\varepsilon + u^2/2 + n(E_r + E_\sigma))_T + (pu)_X\} +$$
$$+ (-\omega(\varepsilon + u^2/2 + n(E_r + E_\sigma)) + kpu)_\theta = -8\nu_1 nE_r/R^2,$$
$$\delta n_T + (-n\omega)_\theta = 0,\ \delta c_{2T} + (-c_2\omega)_\theta = 0.$$

We will also assume that the viscosity ν_1 of the liquid is either zero or $O(\delta)$. In the latter case, $\nu = \lim_{\delta\to 0} \nu_1/\delta$. Writing $\mathbf{w}$ as a series expansion of the form $\mathbf{w} = \mathbf{w}_0(\theta, T, X) + \delta\mathbf{w}_1(\theta, T, X) + \ldots$, we retain only terms of order δ^0 or δ in the expanded system of equations. To order δ^0 we have the conservation laws (after integration with respect to θ)

$$v_0\omega + u_0 k = M(T, X),\ -u_0\omega + p_0 k = P(T, X),$$
$$n_0 = N(X),\ c_{20} = C(X),$$
$$-\omega\left(\varepsilon_0 + n_0(E_{r0} + E_{\sigma 0}) + u_0^2/2\right) + kp_0u_0 = A(T, X),$$

and to order δ we have the equations

$$v_{0T} - u_{0X} + (-v_1\omega - u_1 k)_\theta = 0,\ u_{0T} + p_{0X} + (-u_1\omega + p_1 k)_\theta = 0,$$
$$(\varepsilon_0 + n_0(E_{r0} + E_{\sigma 0}) + u_0^2/2)_T + (p_0u_0)_X + F(\mathbf{w}_0, \mathbf{w}_1)_\theta = -8\nu n_0E_{r0}/R_0^2,$$
$$n_{0T} + (-n_1\omega)_\theta = 0,\ c_{20T} + (-c_{21}\omega)_\theta = 0.$$

The function $F(\mathbf{w}_0, \mathbf{w}_1)$ can easily be recovered from the energy equation. For an arbitrary function $f(\theta, T, X)$ we define

$$\langle f\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(\theta, T, X)\, d\theta.$$

Then, using the periodicity of $\mathbf{w}_0$ and $\mathbf{w}_1$, and integrating with respect to θ, we obtain the following system of equations for $\mathbf{w}_0$ and $\langle\mathbf{w}_0\rangle$ (the subscript 0 on the dependent variables will be omitted):

$$\langle v\rangle_T - \langle u\rangle_X = 0,\ \langle u\rangle_T + \langle p\rangle_X = 0; \tag{2.1}$$
$$\langle\varepsilon + n(E_r + E_\sigma) + u^2/2\rangle_T + \langle pu\rangle_X = -8\nu\langle nE_r/R^2\rangle; \tag{2.2}$$
$$\langle n\rangle_T = 0,\ \langle c_2\rangle_T = 0. \tag{2.3}$$

Since n and c_2 do not depend on θ, we have $\langle n\rangle = n$, $\langle c_2\rangle = c_2$. For simplicity, we will assume that n and c_2 are constants. The equations to order δ^0 can be written in the form

$$v\omega + uk = \langle v\rangle\omega + \langle u\rangle k,\ -u\omega + pk = -\langle u\rangle\omega + \langle p\rangle k; \tag{2.4}$$
$$-\omega(\varepsilon + n(E_r + E_\sigma) + u^2/2) + puk = A(T, X). \tag{2.5}$$

We note that in the case $\nu_1 = 0$, Eqs. (2.1)-(2.5) could have been obtained using the averaged Lagrangian method [7], formulated for the following functional, which represents the action in the sense of Hamiltonian mechanics:

$$I = \int_{t_0}^{t_1}\int_{x_0}^{x_1} [q_t^2/2 + n(E_r - E_\sigma) - \varepsilon + p(q_x - c_1/\rho_1 - n4\pi R^3/3)]\, dt\, dx$$

($q_t = u$, $q_x = v$, q is the Euler coordinate).

The problem then consists of finding a closed system of equations for the average quantities. The system of equations is obtained in two steps. First, we prove the following rather remarkable fact.

THEOREM 2.1. The following equation is satisfied: $\langle pu\rangle = \langle p\rangle\langle u\rangle + O(\alpha_2{}^2)$, $\langle u\rangle^2 = \langle u\rangle^2 + O(\alpha_2{}^2)$ (α_2 is the volumetric concentration of bubbles).

This means that the corresponding correlations are decoupled from one another. Indeed, the original system of equations (1.1)-(1.3) was obtained by discarding terms of order $O(\alpha_2^2)$ [1], and therefore it makes no sense to retain terms of this order in the averages. Second, we write out specific formulas for quantities of the type $\langle E_r\rangle$.

Proof. Multiply the first equation of (2.4) by vk^{-1} and then take the average value. Multiply the same equation by uk^{-1} and then take the average value. Letting $D = \omega/k$, we have

$$\langle uv\rangle - \langle u\rangle\langle v\rangle = (\langle v\rangle^2 - \langle v^2\rangle)D; \tag{2.6}$$

$$\langle u^2\rangle - \langle u\rangle^2 = (\langle v\rangle\langle u\rangle - \langle vu\rangle)D. \tag{2.7}$$

It then follows that

$$\langle u^2\rangle - \langle u\rangle^2 = (\langle v^2\rangle - \langle v\rangle^2)D^2. \tag{2.8}$$

Multiplying the second equation of (2.4) by uk^{-1} and using (2.8), we obtain

$$\langle pu\rangle - \langle p\rangle\langle u\rangle = D^3(\langle v^2\rangle - \langle v\rangle^2). \tag{2.9}$$

Note that $v = c_1/\rho_1 + 4\pi R^3n/3$, i.e., $\langle v^2\rangle - \langle v\rangle^2 = (4\pi n)^2(\langle R^6\rangle - \langle R^3\rangle^2)/9 = O(\alpha_2{}^2)$. Because the phase velocity D is finite [in the opposite case it follows from (2.4) that the dependence on the fast variables drops out], the assertion of the theorem follows from (2.8) and (2.9).

We calculate the quantities $\langle \varepsilon + n(E_r + E_c)\rangle$ and $\langle E_r/R^2\rangle$. To do this we transform (2.5) Substituting the expressions $u = \langle u\rangle - (v - \langle v\rangle)D$, $p = \langle p\rangle - (v - \langle v\rangle)D^2$ in place of u and p, we find $\varepsilon + n(E_r + E_\sigma) + \langle p\rangle v - (v - \langle v\rangle)^2D^2/2 = H(T, X)$, $H(T, X) = -A(T, X)/\omega - \langle u\rangle^2/2 + \langle p\rangle\langle u\rangle/D + \langle p\rangle\langle v\rangle$. Because $(v - \langle v\rangle)^2 = O(\alpha_2{}^2)$, we finally obtain

$$\varepsilon + n(E_r + E_\sigma) + \langle p\rangle v = H(T, X) \tag{2.10}$$

(H can be interpreted as the specific internal energy of the mixture). In particular, it follows from (2.10) that

$$\langle \varepsilon + n(E_r + E_\sigma)\rangle = H - \langle p\rangle\langle v\rangle. \tag{2.11}$$

Because $E_r = 2\pi R^3\rho_1 R_\theta{}^2\omega^2$, (2.10) can be rewritten in the form

$$2\pi R^3\rho_1 n\omega^2 R_\theta^2 = \Phi(H, \langle p\rangle, R) \equiv H - \langle p\rangle v - nE_\sigma - \varepsilon. \tag{2.12}$$

If the function Φ has two simple zeroes: $R_1(H, \langle p\rangle)$, $R_2(H, \langle p\rangle)$ ($R_1 < R_2$) and $\Phi > 0$ between these zeroes, then this will mean that the fast variable has been introduced correctly. An example is given below where this is true. Since the period of the oscillations is 2π, we obtain a nonlinear dispersion relation from (2.12):

$$\omega = \omega(H, \langle p\rangle) = \left(\sqrt{\frac{2}{\pi}n\rho_1}\int_{R_1}^{R_2}\frac{R^{3/2}}{\sqrt{\Phi}}\right)^{-1}. \tag{2.13}$$

Furthermore,

$$\langle nE_r/R^2\rangle = \frac{1}{2\pi}\int_0^{2\pi} nE_r/R^2 d\theta = \frac{1}{\pi}\int_{R_1}^{R_2}\frac{nE_r dR}{R^2 dR/d\theta} = \left(\int_{R_1}^{R_2} R^{-1/2}\sqrt{\Phi}\,dR\right)\Bigg/\left(\int_{R_1}^{R_2}\frac{R^{3/2}dR}{\sqrt{\Phi}}\right), \tag{2.14}$$

and the average specific volume of the mixture is

$$\langle v\rangle = V(H,\langle p\rangle) = c_1/\rho_1 + 4\pi n\langle R^3\rangle/3 = \\ = c_1/\rho_1 + (4\pi n/3)\left(\int_{R_1}^{R_2} \frac{R^{9/2}\,dR}{\sqrt{\Phi}}\right)\Bigg/\left(\int_{R_1}^{R_2}\frac{R^{3/2}dR}{\sqrt{\Phi}}\right). \tag{2.15}$$

Finally, we obtain the following system of equations for the averages $\langle u\rangle$, $\langle p\rangle$, and H in terms of the original unstretched variables x and t using Theorem 2.1 and the relation (2.11):

$$\langle v\rangle_t - \langle u\rangle_x = 0; \tag{2.16}$$

$$\langle u\rangle_t + \langle p\rangle_x = 0; \tag{2.17}$$

$$(H - \langle p\rangle\langle v\rangle + \langle u\rangle^2/2)_t + (\langle p\rangle\langle u\rangle)_x = -8\nu_1\langle nE_r/R^2\rangle \tag{2.18}$$

[$\langle v\rangle = V(H, \langle p\rangle)$ is given by (2.15), and $\langle nE_r/R^2\rangle$ by (2.14)]. If the average quantities are known, we can find the form of the fluctuations. Indeed, from the consistency equation $\theta_{tx} = \theta_{xt}$ it follows that

$$k_t + \omega_x = 0. \tag{2.19}$$

If k(0, x) is specified, then k(t, x) can be found from (2.13) and (2.19). The function R(θ, t, x) is found from (2.12), while u(θ, t, x) and p(θ, t, x) are found from (2.4).

3. Analysis of the System (2.16)-(2.18) and Examples. The modulation equations have the same form as the equations of motion of an inviscid, nonconducting gas, in which there is an energy drain (if $\nu_1 \neq 0$), and where H is the enthalpy of the gas. Below we obtain expressions for the "temperature" and "entropy" of the gas–liquid mixture.

THEOREM 3.1. Let $\tau = \omega$, $S = 2\sqrt{2n\rho_1/\pi}\int_{R_1}^{R_2} R^{3/2}\sqrt{\Phi}\,dR$. Then $\tau dS = dH - \langle v\rangle d\langle p\rangle$.

Proof. It is sufficient to demonstrate that $\tau S_H = 1$, $\tau S_{\langle p\rangle} = -\langle v\rangle$, which can be verified by direct differentiation and use of (2.15).

Note. If $\nu_1 \neq 0$, the "entropy" S (which is a measure of the intensity of the oscillations in the mixture) in the continuous solutions decreases.

We discuss the type of the system of modulation equations. Calculating the slopes of the characteristics, we obtain $\lambda_{1,3} = \mp 1/\sqrt{-VV_H - V_{\langle p\rangle}}$, $\lambda_2 = 0$. Hence the equations are hyperbolic if

$$VV_H + V_{\langle p\rangle} < 0. \tag{3.1}$$

THEOREM 3.2. Let R_* be a point of equilibrium such that $\Phi_R(H, \langle p\rangle, R_*) = 0$. We also suppose that $\Phi_{RR}(H, \langle p\rangle, R_*) < 0$. Then the inequality (3.1) is satisfied for small deviations from the equilibrium position.

Proof. We expand the function $\Phi(H, \langle p\rangle, R)$ in a Taylor series about the point R_*, $\Phi_R(H, \langle p\rangle, R_*) = 0$:

$$\Phi(H,\langle p\rangle,R) = \Phi_* + \Phi_{RR}(H,\langle p\rangle,R_*)(R-R_*)^2/2 + \dots \tag{3.2}$$

Because $\Phi_R = 4\pi R^2 n(p_2 - 2\sigma/R - \langle p\rangle)$, we have

$$\Phi_{RR}(H,\langle p\rangle,R_*) = 4\pi R_* n\,(2\sigma/R_* + 3v_2 dp_2/dv_2|_{R=R_*}). \tag{3.3}$$

In particular, it follows that if the gas making up the bubbles is polytropic and $\langle p\rangle > 0$, then $\Phi_{RR}(H, \langle p\rangle, R_*) < 0$. Indeed, $\operatorname{sgn}\Phi_{RR}(R_*) = \operatorname{sgn}(p_2(1 - 3\gamma) - \langle p\rangle) = -1$, since $\gamma > 1$ (γ is the polytropic exponent). The leading term determining the equation of state is written in the form $v = c_1/\rho_1 + 4\pi n R_*^3/3$. Because R_* depends only on $\langle p\rangle$, we have $V_H = 0$, and the derivative with respect to $\langle p\rangle$ can be calculated from the formula

$$V_{\langle p\rangle} = 4\pi n R_*^2 dR_*/d\langle p\rangle = 4\pi n R_*^3/(2\sigma/R_* + 3v_2 dp_2/dv_2|_{R=R_*}).$$

The sign of the denominator of the fraction in the expression for $V_{\langle p\rangle}$ is the same as that of $\Phi_{RR}(H, \langle p\rangle, R_*)$. Hence the theorem is proved.

We will put $\sigma = 0$ everywhere below. We calculate the leading term of the frequency ω determined by (2.13) for this case, assuming small deviations from the equilibrium position. Let $B = -2\Phi_*/\Phi_{RR}(R_*)$ [see (3.2)]. Then

$$\omega \sim \omega_0 = \left\{ \sqrt{\frac{2}{\pi} n\rho_1} R_*^{3/2} \int\limits_{R_{1*}}^{R_{2*}} \frac{dR}{\sqrt{-\frac{\Phi_{RR}(R_*)}{2}} \sqrt{B-(R-R_*)^2}} \right\}^{-1}$$

[R_{i*} are the roots of the equation $B = (R - R_*)^2$]. Making the substitution $R = (R_{2*} + R_{1*})/2 + s(R_{2*} - R_{1*})/2$ and using (3.3) with $\sigma = 0$, we obtain

$$\omega_0 = \sqrt{\frac{-3v_2 dp_2/dv_2}{\rho_1 R_*^2}}\Bigg|_{R=R*}.$$

If the gas is polytropic, then

$$\omega_0 = \sqrt{3\gamma\langle p\rangle/(\rho_1 R_*^2)}. \qquad (3.4)$$

As one would expect, (3.4) coincides with the Minnaert resonance frequency [4, p. 303].

We consider the equation of state (2.15) for strong oscillations in the case when the gas making up the bubbles is polytropic with $\gamma = 2$. The roots of the equation $\Phi = 0$ can be calculated explicitly for this value of γ. The function Φ can be written in the form

$$\Phi = H - \langle p\rangle (c_1/\rho_1 + 4\pi R^3 n/3) - p_0 R_0^6 4\pi R^{-3} n/3 \qquad (3.5)$$

[p_0 and R_0 are constants characterizing the adiabatic process ($p_2 R^6 = p_0 R_0{}^6$)]. Let

$$a = H - \langle p\rangle c_1/\rho_1,\ b = 4\pi n\langle p\rangle/3,\ c = 4\pi n R_0^6 p_0/3,\ z = R^3. \qquad (3.6)$$

Then (2.15) is transformed to

$$\langle v\rangle = V(H, \langle p\rangle) = c_1/\rho_1 + (4\pi n/3) I(4/3)/I(1/3) \qquad (3.7)$$

[$I(\alpha) = \int\limits_{z_1}^{z_2} \frac{z^\alpha\, dz}{\sqrt{az - bz^2 - c}}$, z_i are the roots of the equation $az - bz^2 - c = 0$]. Making the substitution $z = z_1 + t(z_2 - z_1)$, we have

$$I(\alpha) = \frac{(z_2 - z_1)^\alpha}{\sqrt{b}} \int\limits_0^1 \frac{(t + z_1/(z_2 - z_1))^\alpha\, dt}{\sqrt{t(1-t)}}.$$

We assume strong oscillations, i.e., the parameter $\varepsilon_1 = z_1/(z_2 - z_1)$ is small. Then it follows from (3.7) that

$$\langle v\rangle = c_1/\rho_1 + 4\pi n(z_2 - z_1)/3 \frac{\int\limits_0^1 t^{5/6}(1-t)^{-1/2}\, dt + m_1}{\int\limits_0^1 t^{-1/6}(1-t)^{-1/2}\, dt + m_2}$$

[$m_1 = O(\varepsilon_1)$, $m_2 = O(\varepsilon_1{}^{1/3})$]. An estimate for m_1 follows from the inequality $(t + \varepsilon_1)^\alpha - t^\alpha \leq \varepsilon_1 \alpha (t + \varepsilon_1)^{\alpha-1}$ $(1 < \alpha < \infty)$ and an estimate for m_2 follows from $(t + \varepsilon_1)^{1/3} - t^{1/3} \leq \varepsilon_1{}^{1/3}$. Because $z_1 = \varepsilon_1 z_2 + O(\varepsilon_1{}^2)$, we finally obtain, with the help of (3.6),

$$\langle v\rangle = c_1/\rho_1 + \mu (H - \langle p\rangle c_1/\rho_1)/\langle p\rangle + O(\varepsilon_1^{1/3}). \qquad (3.8)$$

Here the coefficient μ is calculated from the equation

$$\mu = \mathrm{B}(11/6,\ 1/2)/\mathrm{B}(5/6,\ 1/2) \qquad (3.9)$$

(B(x, y) is the complete beta function [9]). Since $B(p, q) = \Gamma(p)\Gamma(q)/\Gamma(p + q)$ [$\Gamma(z)$ is the gamma function, $\Gamma(z + 1) = z\Gamma(z)$], it follows from (3.9) that $\mu = 5/8$. Since $\mu < 1$, the inequality (3.1) is satisfied, i.e., the system of modulation equations is hyperbolic. Letting

$$H \to H + \langle p\rangle c_1/\rho_1, \ \langle v\rangle \to \langle v\rangle + c_1/\rho_1 \tag{3.10}$$

and discarding terms of order $O(\varepsilon_1{}^{1/3})$, we obtain from (3.8) that the gas–liquid medium behaves as a polytropic gas with index 8/3. It is interesting to note that this "gas" is not an ideal gas [5]. To show this we calculate the "entropy" S (see Theorem 3.1):

$$S = \sqrt{\frac{2}{\pi} n\rho_1} \frac{2}{3} \sqrt{b}\,(z_2 - z_1)^{4/3} \int_0^1 (t + z_1/(z_2 - z_1))^{-2/3} \sqrt{t(1-t)}\,dt =$$

$$= \sqrt{\frac{2}{\pi} n\rho_1} \frac{2}{3} \sqrt{b} \left(\frac{a}{b}\right)^{4/3} \int_0^1 (1-t)^{1/2} t^{-1/6} dt + m_3$$

[$m_3 = O(\varepsilon_1{}^{1/3})$]. An estimate for m_3 follows from the inequality $|(t + \varepsilon_1)^{-2/3} - t^{-2/3}| \le 2\varepsilon_1{}^{1/3}t^{-1}$. Or, discarding terms of order $O(\varepsilon_1{}^{1/3})$, and using (3.6) and (3.10), we have

$$S = c_0 H^{4/3} \langle p\rangle^{-5/6}, \ c_0 = \sqrt{2n\rho_1/\pi}(4\pi n/3)^{-5/6}(2/3)B(5/6, 3/2). \tag{3.11}$$

And from (3.8)-(3.11) we find

$$\langle p\rangle = \left(\frac{S}{c_0}\right)^2 \mu^{8/3} \langle v\rangle^{-8/3} = \frac{c_0^2}{(3/4)^2 \mu^{2/3}} \langle v\rangle^{2/3} \tau^2$$

[$\tau = (S_H)^{-1}$ is the "temperature"). Hence the Clapeyron equation does not hold.

4. Discontinuous Solutions of the Modulation Equations. It is evident from the above examples that the system of modulation equations is hyperbolic in a certain region of the parameter space. Hence, one can consider shock waves in the medium. We note that shock waves cannot exist in the original system of equations (1.1)-(1.3) because the propagation velocity of weak perturbations is infinite and, therefore, the supersonic nature of the motion ahead of the front of the shock wave cannot be described by these equations.

The Hugoniot conditions for (2.16)-(2.18) are the usual conservation laws of mass, momentum, and energy across the discontinuity. The frequency ω behind the front is given by the final equation of (2.13). Finally, the constancy of the phase across the jump, which follows from (2.19), $U[k] = [\omega]$ (U is the velocity of the discontinuity and the square brackets denote the jump of the corresponding quantity) gives the value of the wave number behind the front. Hence, the system of equations is complete on the discontinuity.

The question of the constancy of the phase is controversial. Some reasons for constancy of phase, and also some alternative approaches, are given in [7, pp. 500-503; 10, pp. 42-45].

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THE WAVEGUIDE EFFECT

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The main purpose of scattering theory is the study of qualitative features of scattered waves. In the present study we investigate anomalous effects of the type of the waveguide effect for scattering problems by one-dimensional periodic structures. According to the definition of R. M. Garipov, the waveguide effect consists of the existence of eigenwaves localized in the vicinity of the structure. The properties of these waves are described by generalized eigenfunctions, being solutions of problems for steady-state oscillations. We consider existence conditions and the possibility of a waveguide effect for one-dimensional periodic structures: for long waves on shallow water – a one-dimensional periodic underwater ridge of the plateau type; and for acoustic or electromagnetic waves – a one-dimensional periodic lattice of plates or smooth obstacles.*

1. Formulation of the Problem. Required Information. Let Γ describe on the plane $\mathbf{R}^2$ of Cartesian variables (x, y) the boundary between free space and an obstacle. It is assumed that Γ can be connected by a curve or a set of quite smooth closed or disconnected curves. It is assumed that Γ is periodic along the y axis with period 2π. The obstacle can be penetrable or impenetrable (Fig. 1).

Wave effects near the obstacle are described by a quite smooth complex function u(x, y) outside the obstacle boundary Γ, whose physical content is specific to the problem. Let Ω_1 and Ω_2 be the regions into which Γ divides the plane $\mathbf{R}^2$. The contraction of the function u(x, y) to regions Ω_1 and Ω_2 is denoted by $u_1(x, y)$ and $u_2(x, y)$, respectively. The functions $u_1(x, y)$ and $u_2(x, y)$ must be solutions of the Helmholtz equation:

$$(\Delta + \varkappa^2\lambda^2)u_1 = 0 \text{ in } \Omega_1,\ (\Delta + \lambda^2)u_2 = f \text{ in } \Omega_2. \tag{1.1}$$

The following matching conditions are satisfied on the boundary Γ of the regions Ω_1 and Ω_2:

$$\delta u_1 = u_2,\ \gamma\partial u_1/\partial \mathbf{n} = \partial u_2/\partial \mathbf{n} \text{ on } \Gamma. \tag{1.2}$$

Here $\varkappa > 0$, $\delta > 0$, $\gamma > 0$ are real, and λ is a complex parameter, whose physical meaning is determined by the content of the effect investigated. The function f(x, y) describes sources of oscillation, and is assumed periodic in y with period 2π and localized in the vicinity of the structure. All functions satisfy the condition of local finite energy $[u_1 \in W_{2\,loc}^1(\Omega_1),\ u_2 \in W_{2\,loc}^1(\Omega_2)]$, and are assumed periodic along the y axis with period 2π.

The general solution of the homogeneous Helmholtz equation with parameter λ, satisfying the periodicity condition along y with period 2π, is

$$u(x, y) = \sum_{k=-\infty}^{+\infty} [a_k^{\pm} \exp(iky + i|x|\sigma_k) + b_k^{\pm} \exp(iky - i|x|\sigma_k)], \tag{1.3}$$

*The basic results of this study were presented at the 6th All-Union Congress on Theoretical and Applied Mechanics.

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